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A note on the wall-jet problem

J.H. MERKIN [1] and D.J. NEEDHAM [2]

[1] *School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K.*
[2] *School of Mathematics and Physics, University of East Anglia, Norwich NR4 7TJ, U.K.*

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Summary

The wall-jet problem is considered with, in turn, the effects of blowing and suction through the wall and the wall moving. In both cases it is assumed that the effect has the appropriate power-law variation so as to maintain the similarity form. It is shown that the resulting ordinary differential equation has no acceptable solution satisfying the required boundary conditions. The wall jet problem in which the original momentum condition is retained, but allowing for both transpiration velocity and the wall velocity is derived and it is shown that this implies that a solution is possible only for suction. A solution to this problem is then obtained for the appropriate power-law variations to keep the similarity form.

1. Introduction

The plane wall-jet solution of the boundary-layer equations was given by Glauert [1,2], the problem arising when a jet of fluid is forced along a wall surrounded by fluid of the same type and otherwise at rest. Glauert [1], showed that the boundary-layer equations could be reduced by a similarity transformation and was able to give a closed-form solution of the resulting ordinary differential equation. The effects of compressibility on the wall jet were treated later by Riley [3].

Here we consider again the wall-jet problem and allow, in turn, for the effects of blowing or suction through the wall, and the wall moving. In both cases we assume that the effect has the appropriate power-law variation in x (x is non-dimensional distance along the wall) for the similarity form, given in [1], to be preserved. For both of the above cases we show that the modified similarity problems have no solution.

For the two problems mentioned above the momentum condition which drives Glauert's original wall-jet solution has to be relaxed. We then go on to consider the situation in which this condition is retained but allow as well for the effects of transpiration and the wall moving. We show that these two effects are now connected through an integral condition, which implies that a solution is possible only for suction, not for blowing. We then obtain the corresponding similarity solution and describe its properties for large suction velocities.

2. Transpiration and moving-wall cases

Consider first the case of suction or blowing through the wall. The transformation given in [1] requires a transpiration velocity $v_w(x) = -\frac{1}{4}\alpha x^{-3/4}$ to maintain similarity. The resulting ordinary differential equation is then, with $\psi = x^{1/4}f(\eta)$, $\eta = y/x^{3/4}$ where y measures distance normal to the wall and x along it,

$$f''' + \tfrac{1}{4}ff'' + \tfrac{1}{2}f'^2 = 0 \tag{1}$$

with boundary conditions

$$f(0) = \alpha, \quad f'(0) = 0, \quad f' \to 0 \quad \text{as} \quad \eta \to \infty, \tag{2}$$

where a prime denotes differentiation with respect to η.

We now show that the only solution of equation (1) that satisfies (2) is the trivial solution $f = \alpha$ for any non-zero α. To do this, first put $\phi = f - \alpha$ (so $\phi(0) = 0$), and then multiply the resulting equation by ϕ. This gives

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\phi\phi'' - \tfrac{1}{2}\phi'^2 + \tfrac{1}{4}\phi^2\phi' + \tfrac{1}{4}\alpha\phi\phi'\right) - \tfrac{1}{4}\alpha\phi'^2 = 0. \tag{3}$$

Then, by integrating equation (3) with respect to η from $\eta = 0$ to ∞ and using the conditions at $\eta = 0$ and as $\eta \to \infty$, we have

$$\int_0^\infty \phi'^2 \, \mathrm{d}\eta = 0, \tag{4}$$

from which it follows that $\phi \equiv 0$, so the only solution of (1) and (2) is the trivial solution $f \equiv \alpha$. The non-similar problem of the wall jet with constant suction through the wall has been discussed by Elliott and Watson [4], who showed that a weak singularity developed at a finite value of x and were able to deduce the nature of this singularity.

Next consider the effect of a moving wall. To maintain the similarity form we need a wall velocity $U_w(x)$ of the form $U_w(x) = \beta x^{-1/2}$, and, with this wall velocity, the boundary-layer equations again reduce to (1) but now with boundary conditions

$$f(0) = 0, \quad f'(0) = \beta, \quad f' \to 0 \quad \text{as} \quad \eta \to \infty. \tag{5}$$

In this case also, we can show that equation (1) has no solution which satisfies (5). On multiplying equation (1) by f and integrating once we find

$$ff'' - \tfrac{1}{2}f' + \tfrac{1}{4}f^2f' = C \tag{6}$$

where C is a constant. The boundary conditions on $\eta = 0$ imply that $C = -\tfrac{1}{2}\beta^2$, whereas the boundary conditions as $\eta \to \infty$ require $C = 0$. These are incompatible for $\beta \neq 0$. Equation (6) with boundary conditions (5) is a particular example of a class of equations discussed by Merkin [5] and Banks [6].

3. Retaining the momentum condition

The (non-dimensional) momentum equation governing the flow in the boundary layer is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}, \tag{7}$$

which, together with the continuity equation, has to be solved subject to the boundary conditions

$$v = v_w(x), \quad u = U_w(x) \quad \text{on} \quad y = 0, \quad u \to 0 \quad \text{as} \quad y \to \infty. \tag{8}$$

On integrating equation (7) and using the continuity equation, we obtain the condition

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \tfrac{1}{2}U_w^2 + v_w \int_0^\infty u^2 \, \mathrm{d}y \tag{9}$$

where

$$M = \int_0^\infty u\left(\int_y^\infty u^2 \, \mathrm{d}\bar{y}\right) \mathrm{d}y.$$

The two previous cases discussed above have either $U_w = 0$ or $v_w = 0$, which through (9) gives a connection between M and either v_w or U_w. If we now insist on retaining the momentum condition as Glauert did in his original problem, i.e. taking M to be a constant (in fact we can take $M = 1$ without loss in generality by defining the non-dimensional variables suitably), we have the condition

$$\tfrac{1}{2}U_w^2 + v_w \int_0^\infty u^2 \, \mathrm{d}y = 0 \tag{10}$$

connecting v_w and U_w. From (10) it is easy to see that the only possibility is to have $v_w < 0$, i.e. we can allow only suction through the wall.

If we now take the forms for v_w and U_w as required for a similarity solution (i.e. $v_w = -\tfrac{1}{4}\alpha x^{-3/4}$ and $U_w = \beta x^{-1/2}$) we then have to solve equation (1) subject to

$$f(0) = \alpha, \quad f'(0) = \beta, \quad f' \to 0 \quad \text{as} \quad \eta \to \infty \tag{11}$$

and the momentum condition (taking $M = 1$)

$$\int_0^\infty (f - \alpha) f'^2 \, \mathrm{d}\eta = 1 \tag{12}$$

with α and β connected by the relation

$$2\beta^2 = \alpha \int_0^\infty f'^2 \, \mathrm{d}\eta. \tag{13}$$

Following [1], equation (1) can be integrated twice to give

$$f' = \tfrac{1}{6} f^{1/2}(\sigma^3 - f^{3/2}) \tag{14}$$

where $\sigma^2 = f(\infty)$ is some as yet unknown constant which depends on α and will be determined by the integral condition (12). Equation (14) can be integrated once more to give f implicitly in terms of η as

$$\log\left(\frac{(f+\sigma f^{1/2}+\sigma^2)(\sigma-\alpha^{1/2})^2}{(\alpha+\sigma\alpha^{1/2}+\sigma^2)(\sigma-f^{1/2})^2}\right)+2\sqrt{3}\,\tan^{-1}\left(\frac{2\sqrt{3}\,\sigma(f^{1/2}-\alpha^{1/2})}{1+(2f^{1/2}+\sigma)(2\alpha^{1/2}+\sigma)}\right)$$

$$=\frac{\sigma^2}{2}\eta. \tag{15}$$

To determine the relation between σ and α it is easiest to use equation (14) directly in (12); this gives the equation

$$\sigma^8-\tfrac{20}{9}\alpha\sigma^6+\tfrac{16}{9}\alpha^{5/2}\sigma^3-\left(40+\frac{5\alpha^4}{9}\right)=0. \tag{16}$$

On putting $\sigma=\mu\alpha^{1/2}$ $(\alpha>0)$, equation (16) can be written as

$$(\mu-1)^3 g(\mu)-40\alpha^{-4}=0 \tag{17}$$

where

$$g(\mu)=\mu^5+3\mu^4+\tfrac{34}{9}\mu^3+\tfrac{10}{3}\mu^2+\tfrac{5}{3}\mu^2+\tfrac{5}{9}.$$

It is then easy to deduce, since $g(\mu)>0$ and $g'(u)>0$ for all $\mu\geqslant 0$, that equation (17) has

Table 1. Values of σ obtained by solving equation (16) compared with the asymptotic solution given by (19), and the corresponding values of β

α	From (16)	From (19)	β
0	1.58583		0.00000
0.5	1.68032		0.51746
1.0	1.78187		0.77625
1.5	1.88554		0.99335
2.0	1.98897		1.18792
2.5	2.09096		1.36744
3.0	2.19090		1.53582
3.5	2.28851		1.69547
4.0	2.38368		1.84798
4.5	2.47643		1.99449
5.0	2.56681	2.61326	2.13585
6.0	2.74083	2.77352	2.40566
7.0	2.90658	2.93072	2.66123
8.0	3.06489	3.08338	2.90518
9.0	3.21654	3.23112	3.13937
10.0	3.36220	3.37397	3.36519
12.0	3.63787	3.64596	3.79584
15.0	4.01889	4.02398	4.39998
20.0	4.58817	4.59094	5.32489
25.0	5.09692	5.09865	6.17556
30.0	5.56080	5.56197	6.97128

just one root in $\mu > 0$ and hence for a given value of α there corresponds a unique value of σ. Thus to obtain a solution for a given value of α we first solve equation (16) to determine σ and then use this value in either equation (14) or (15) to determine f. The corresponding value of β given by equation (14) is then

$$\beta = \tfrac{1}{6}\alpha^{1/2}(\sigma^3 - \alpha^{3/2}), \tag{18}$$

and it is straightforward to check that this is consistent with β as determined by (13).

Equation (17) enables us to determine the solution for large α; we find that

$$\mu = 1 + 3^{1/3}\alpha^{-4/3} + \ldots \tag{19}$$

from which it follows, from (15) or equation (14), that for $\alpha \gg 1$

$$f = \alpha + 2.3^{1/3}(1 - e^{-\zeta/4})\alpha^{-1/3} + \ldots \tag{20}$$

where $\zeta = \alpha\eta$.

Values of σ obtained by solving equation (16) numerically using a Newton-Raphson method are given in Table 1 for a range of α. Also shown in this table are the corresponding values of β as given by (18) and the values of α calculated from the asymptotic expression (19). Graphs of f' for various values of α obtained by solving equation (14) numerically using the appropriate value of σ (as given in Table 1) are shown in Figure 1.

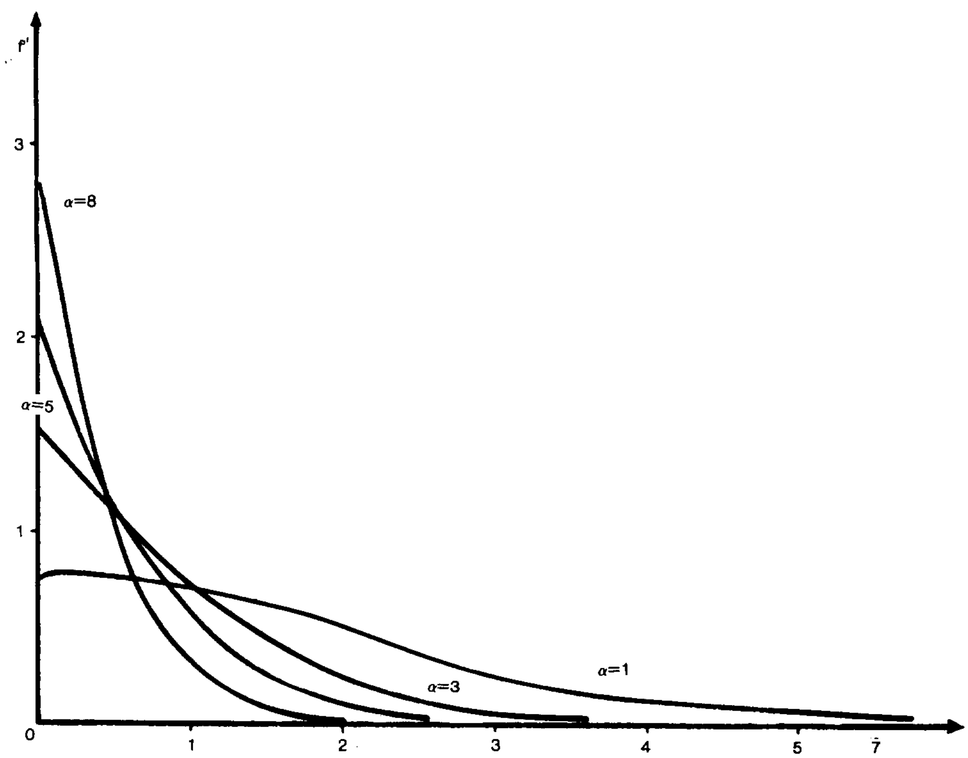

Figure 1. Graphs of f' for various values of α.

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